

Small $(2, s)$ -colorable graphs without 1-obstacle representations*

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Abstract

An obstacle representation of a graph G is a set of points on the plane together with a set of polygonal obstacles that determine a visibility graph isomorphic to G . The obstacle number of G is the minimum number of obstacles over all obstacle representations of G .

Alpert, Koch, and Laison [1] gave a 12-vertex bipartite graph and proved that its obstacle number is *two*. We show that a 10-vertex induced subgraph of this graph has obstacle number *two*.

Alpert et al. [1] also constructed very large graphs with vertex set consisting of a clique and an independent set in order to show that obstacle number is an unbounded parameter. We specify a 70-vertex graph with vertex set consisting of a clique and an independent set, and prove that it has obstacle number greater than *one*.

This is an ancillary document to our article in press [8]. We conclude by showing that a 10-vertex graph with vertex set consisting of two cliques has obstacle number greater than *one*, improving on a result therein.

1 Introduction

Consider a finite set P of points on the plane, and a set of closed polygonal obstacles whose vertices together with the points in P are in general position, that is, no three of them are collinear. The corresponding visibility graph has P as its vertex set, two points $p, q \in P$ having an edge between them if and only if the line segment pq does not meet any obstacles. Visibility graphs are extensively studied and used in computational geometry and robot motion planning; see [3, 5, 6, 7, 10].

Relatively recently, Alpert, Koch, and Laison [1] introduced an interesting new parameter of graphs, closely related to visibility graphs. Given a graph G , we say that a set of points and a set of polygonal obstacles as above constitute an *obstacle representation* of G , if the corresponding visibility graph is isomorphic to G . A representation with h obstacles is called an h -obstacle representation. The smallest number of obstacles in an obstacle representation of G is called the *obstacle number* of G .

A graph is called (r, s) -colorable [2] if its vertex set can be partitioned into r sets, s of which are cliques and $r - s$ of which are independent sets. For instance, $(2, 0)$ -colorable graph is simply a bipartite graph, a $(2, 1)$ -colorable graph is a split graph [4, 9], and a $(2, 2)$ -colorable graph has bipartite complement.

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In our paper in press [8], we employed extremal graph theoretic methods to show that for every constant h , the number of graphs on n vertices with obstacle number at most h is $2^{o(n^2)}$, based on the graphs G_1 , G_2 , and G_3 with the properties stated in the following. In this ancillary note to that paper, we accomplish three tasks. We show that a particular 10-vertex $(2,0)$ -colorable (i.e., bipartite) graph G'_1 has obstacle number greater than *one*. This improves upon the 12-vertex bipartite graph G_1 in [1], and settles a conjecture therein. We also show that a particular 70-vertex $(2,1)$ -colorable graph G'_2 has obstacle number greater than *one*, improving on the $(92379 + \binom{92379}{6})$ -vertex graph implied by a construction in [1]. In [8], we had given a $(2,2)$ -colorable 20-vertex graph G_3 , and proved that it has obstacle number greater than *one*. We finally show that a related $(2,2)$ -colorable 10-vertex graph G'_3 also has obstacle number greater than *one*.

2 A 10-vertex bipartite graph without a 1-obstacle representation

Given a graph, we refer to a distinct pair of vertices of the graph that does not define an edge of the graph as a non-edge. In every drawing of a simple finite graph there is bound to be a unique unbounded face, referred to as the outside face. A 1-obstacle representation in which the obstacle lies on the outside face is called an outside obstacle representation, and such an obstacle is called an outside obstacle.

In [1], $K_{m,n}^*$ has been defined as the graph obtained from the complete bipartite graph $K_{m,n}$ by removing a maximum matching. There, it was shown that every $K_{m,n}^*$ graph admits a 2-obstacle representation: The two independent sets are placed within disjoint half-planes, such that the non-edges in the removed matching meet at a single point so that a single non-outside obstacle is sufficient to meet them, while the non-edges within the independent sets meet the outside face so that an outside obstacle is sufficient to meet them. The authors also gave a strong hint for obtaining an outside obstacle representation of $K_{4,n}^*$ for every n by providing an easily generalizable outside obstacle representation for $K_{4,5}^*$. Furthermore, they proved that $G_1 := K_{5,7}^*$ does not admit a 1-obstacle representation.

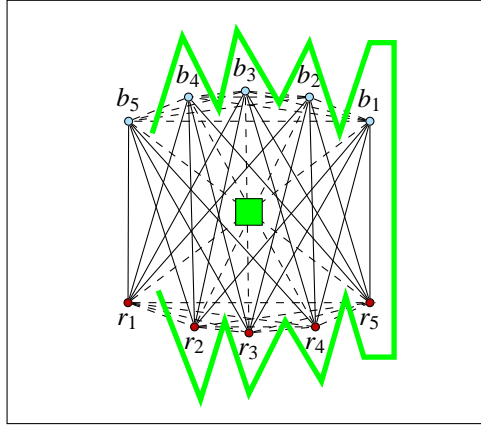


Figure 1: A 2-obstacle representation of G'_1 , i.e., $K_{5,5}^*$.

We dedicate the rest of this section to proving their following conjecture.

Theorem 2.1. $G'_1 := K_{5,5}^*$, the graph obtained from $K_{5,5}$ by removing a perfect matching, has obstacle number 2.

Proof. To able to refer to individual vertices of $K_{5,5}^*$, let $V(K_{5,5}^*) = B \uplus R$ such that $B = \{b_1, b_2, b_3, b_4, b_5\}$ (the set of light blue vertices) and $R = \{r_1, r_2, r_3, r_4, r_5\}$ (the set of dark red vertices) are independent sets and there is an edge from a blue vertex b_i to a red vertex r_j if and only if $i \neq j$.

Before we proceed, we borrow some definitions and two facts from [1].

Given points a, b, c in the plane we say a sees b to the left of c (equivalently, sees c to the right of b) if the points a, b , and c appear in clockwise order. If a point a is outside the convex hull of some set S of points, the relation “ a sees to the left of” is transitive on S , hence is a total ordering of S , called the a -sight ordering of S .

We paraphrase Lemma 3 of [1] in the following way.

Lemma 2.2. *If a graph having $K_{2,3}$ as an induced subgraph has a 1-obstacle representation, then in such a representation the two parts (independent sets) of the induced $K_{2,3}$ are linearly separable. Moreover, for each part S , every vertex in the other part induces the same sight ordering of S .*

Lastly, we paraphrase a fact used in the original proof of Lemma 2.2.

Lemma 2.3. *In a 1-obstacle representation of $K_{5,5}^*$, every vertex subset S consisting of 2 red vertices and 2 blue vertices with 4 distinct subscripts (the necessary and sufficient condition for a $K_{2,2}$ to be induced) is in convex position, with both color classes appearing contiguously around the convex hull of S . Hence the drawing induced on S (i.e., the drawing of every induced $K_{2,2}$ in $K_{5,5}^*$) is self-intersecting, a bowtie.*

We now give and prove a new lemma, one of many to help prune the space of vertex arrangements potentially amenable to 1-obstacle representations of $K_{5,5}^*$.

For any three points p, q, r , we denote by $\angle pqr$ the union of the rays \overrightarrow{qp} and \overrightarrow{qr} . We denote by $\text{conv}(P)$ for the convex hull of a point set P .

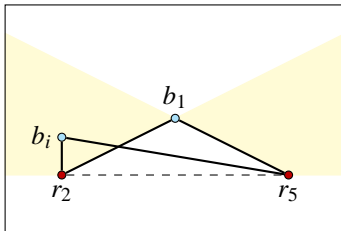
Lemma 2.4. *Every 1-obstacle representation of $K_{5,5}^*$ is an outside obstacle representation.*

Proof. Assume that we are given a 1-obstacle representation of $K_{5,5}^*$ that is not an outside obstacle representation. At least three vertices are on the convex hull boundary of the vertices by the general position assumption. Every pair of vertices appearing consecutively around the convex hull boundary must constitute an edge, otherwise an outside obstacle would be required to block it. Then without loss of generality b_1, r_2, b_3 appear consecutively on the bounding polygon. All other vertices including r_4 are inside $\text{conv}(\angle b_1 r_2 b_3)$. Hence the drawing of the $K_{2,2}$ induced on $\{b_1, r_2, b_3, r_4\}$ is not a bowtie, which contradicts Lemma 2.3. \square

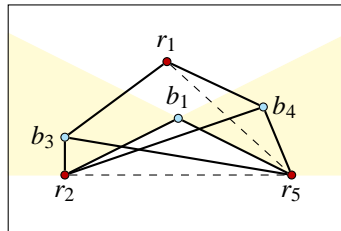
Lemma 2.5. *In every 1-obstacle representation of $K_{5,5}^*$, every vertex v is linearly separable from the set S of its neighbors, defining a v -sight ordering on S .*

Proof. Assume for contradiction that we are given a 1-obstacle representation of $K_{5,5}$ in which some vertex, without loss of generality, b_1 , is not linearly separable from the set of its neighbors. Then b_1 is in the convex hull of $\{r_2, r_3, r_4, r_5\}$. By the general position assumption, a triangulation of $\{r_2, r_3, r_4, r_5\}$ will reveal that b_1 is inside some triangle with red vertices. Without loss of generality, $\Delta r_3 r_4 r_5$. Then by the general position assumption, the ray $\overrightarrow{b_1 b_2}$ meets an interior point of some edge of this triangle. Without loss of generality, $\overline{r_4 r_5}$. This implies that the drawing of $K_{2,2}$ induced on $\{b_1, r_4, b_2, r_5\}$ is not a bowtie, which contradicts Lemma 2.3. \square

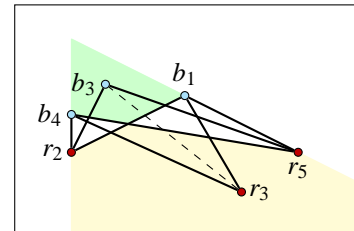
In a graph drawing or obstacle representation, we say that a polygon is *solid* if it is a subset of the drawing: if every point on it is a vertex or on an edge.



(a) For every $i \in \{3, 4\}$, b_i must be in the shaded region.



(b) If b_3 and b_4 are on opposite quadrants of b_1 as shown, the non-edge $r_1 r_5$ cannot be blocked by the outside face.



(c) The “possibility regions” of b_3 and r_3 are shown in different hues. Even if both regions are unbounded, the non-edge $r_3 b_3$ cannot be blocked by the outside face.

Figure 2: Subfigures (a), (b), and (c) respectively accompany the second, third, and last paragraphs in the proof of 2.6. Some edges and non-edges are omitted for clarity, as they often will be in subsequent figures.

Lemma 2.6. *In every 1-obstacle representation of $K_{5,5}^*$, every vertex in R (respectively, B) is linearly separable from B (respectively, R).*

Proof. We will show that in every 1-obstacle representation of $K_{5,5}^*$, each blue vertex is linearly separable from R . The analogous statement about each red vertex and B can be proved symmetrically.

Assume for contradiction that we are given a 1-obstacle representation of $K_{5,5}^*$ in which some blue vertex is in $\text{conv}(R)$. Without loss of generality, $b_1 \in \text{conv}(R)$. By Lemma 2.5, b_1 is linearly separable from $\{r_2, r_3, r_4, r_5\}$. Without loss of generality, $\overleftrightarrow{r_2 r_5}$ is a horizontal line with r_2 to the left of r_5 such that b_1 is above $\overleftrightarrow{r_2 r_5}$ and $r_3 r_4$ is inside $\text{conv}(\angle r_2 b_1 r_5)$. Call the four open regions delineated by the lines $\overleftrightarrow{r_2 b_1}$ and $\overleftrightarrow{r_5 b_1}$ the left, right, upper, and lower b_1 -quadrants. For each $i \in \{3, 4\}$, since the drawing of $K_{2,2}$ induced on $\{r_2, b_i, r_5, b_1\}$ must be a bowtie by Lemma 2.3, b_i is above $\overleftrightarrow{r_2 r_5}$ and in either the left or the right b_1 -quadrant. (See Fig. 2(a).)

Without loss of generality, b_3 is in the left b_1 -quadrant. Assume for contradiction that b_4 is in the right b_1 -quadrant. Since $b_1 \in \text{conv}(R)$, r_1 is in the upper b_1 -quadrant. Then b_3, r_1, b_4 are respectively in the left, upper, and right b_1 -quadrants, and r_5 is on the boundary of the right and lower b_1 -quadrants. This implies that the drawing of $K_{2,2}$ induced on $\{b_3, r_1, b_4, r_5\}$ is non-self-intersecting, not a bowtie. By Lemma 2.3, this means b_4 is in the left b_1 -quadrant along with b_3 . (See Fig. 2(b).)

Notice that $K_{2,3}$ is induced on $\{r_2, r_5, b_1, b_3, b_4\}$. Then by Lemma 2.2, b_3 and b_4 are above $\overleftrightarrow{r_2 r_5}$ along with b_1 , and the r_2 - and r_5 -sight orderings of $\{b_1, b_3, b_4\}$ are the same, with b_1 appearing rightmost. Without loss of generality, r_2 and r_5 see b_4 to the left of b_3 . Hence, b_3 is inside $\text{conv}(\angle b_4 r_2 b_1)$ in addition to being inside $\text{conv}(\angle b_4 r_5 b_1)$. By the same token, since b_1 sees r_3 to be between r_2 and r_5 , so does b_4 . Hence, r_3 is inside $\text{conv}(\angle r_2 b_4 r_5)$, in addition to being inside $\text{conv}(\angle r_2 b_1 r_5)$. These conditions ensure that $\text{conv}(\angle r_2 b_3 r_5)$ and $\text{conv}(\angle b_4 r_3 b_1)$ meet to give a convex quadrilateral region with solid boundary that has $\overline{b_3 r_3}$ as a diagonal. This implies that the non-edge $b_3 r_3$ is not blocked by the outside face, in contradiction to Lemma 2.4. (See Fig. 2(c).) \square

Denote by $K_{3,3}^-$ the graph obtained by removing an edge from $K_{3,3}$. Note that our proof of Lemma 2.6 relies on showing that the assumptions lead to a drawing of $K_{2,2}$ forbidden by Lemma 2.3, or to a forbidden drawing of $K_{3,3}^-$ like the one shown in Fig. 2(c).

Lemma 2.7. *In every 1-obstacle representation of $K_{5,5}^*$, the convex hulls of R and B are disjoint, hence, there is a line separating R from B .*

Proof. Assume for contradiction that we are given a 1-obstacle representation of $K_{5,5}^*$ in which $\text{conv}(R) \cap \text{conv}(B) \neq \emptyset$. Let X denote $\text{conv}(R) \cap \text{conv}(B)$. But by Lemma 2.6, $(R \cup B) \cap X = \emptyset$. This means that X is a $2k$ -gonal shape ($2 \leq k \leq 5$) separating $\text{conv}(B)$ and $\text{conv}(R)$ into k pieces each, alternating around it.

If $k \geq 3$, Without loss of generality, $r_1 b_{i_2} r_2 b_{i_1} r_3 b_{i_3}$ is a counterclockwise enumeration of some convex hexagon H . Take $\overleftrightarrow{r_2 r_3}$ as horizontal. Without loss of generality, b_4 is below $\overleftrightarrow{r_2 r_3}$. By Lemma 2.2, b_1 and b_5 are also below $\overleftrightarrow{r_2 r_3}$. This means that $\{b_{i_2}, b_{i_3}\} = \{b_2, b_3\}$. If $i_2 = 3$, then H is solid and has the non-edge $\overline{b_2 b_3}$ as an internal diagonal, which therefore requires an internal obstacle, contradicting 2.4. Otherwise, $i_2 = 2$ and $\overline{r_2 b_3}$ meets $\overline{b_2 r_3}$ at some point q , so the solid convex quadrilateral $r_1 b_2 q b_3$ has $\overline{b_2 b_3}$ as an internal diagonal, which once again requires an internal obstacle, contradicting 2.4.

Therefore, X separates $\text{conv}(R)$ and $\text{conv}(B)$ into 2 pieces each. Denote by R_1 and R_2 the subsets of R induced by this partition, and define B_1 and B_2 similarly. Without loss of generality, $|R_1| \in \{1, 2\}$ and $|B_1| \in \{1, 2\}$. Now we will show that $|R_1| = |B_1| = 1$.

Assume otherwise for contradiction. Without loss of generality, $R_1 = \{r_1, r_2\}$ and $R_2 = \{r_3, r_4, r_5\}$. By Lemma 2.2, $\overline{r_1 r_3}$ is linearly separable from $\Delta b_2 b_4 b_5$. Clearly, $\overleftrightarrow{r_1 r_3}$ separates B_1 from B_2 . This implies that $\{b_2, b_4, b_5\} \subseteq B_2$. Similarly, $\overline{r_2 r_4}$ is linearly separable from $\Delta b_1 b_3 b_5$, which implies $\{b_1, b_3, b_5\} \subseteq B_2$. But then we have $|B_2| = 5$, a contradiction.

Without loss of generality, let $R_1 = \{r_1\}$. To see that this forces $B_1 = \{b_1\}$, assume for contradiction that (without loss of generality) $B_1 = \{b_2\}$. Then $\overline{r_1 r_3}$ meets $\overline{b_2 b_4}$, contradicting Lemma 2.3.

Without loss of generality, the sets R_1, B_1, R_2, B_2 appear clockwise around X , in this order. Notice that every red vertex in R_2 sees b_1 rightmost in B . Without loss of generality, let the b_1 -sight ordering of R be r_5, r_4, r_3, r_2, r_1 . To highlight the resemblance to the proof of Lemma 2.6, take the line $\overleftrightarrow{r_2 r_5}$ to be horizontal with r_2 to the left of r_5 .

Since $K_{2,3}$ is induced on $\{b_1, b_3, b_4, r_2, r_5\}$, by Lemma 2.2 the r_2 - and r_5 -sight orderings of $\{b_4, b_3, b_1\}$ are the same. Since r_2 and r_5 are in R_2 , they see b_1 as the rightmost blue vertex and without loss of generality they see b_4 to the left of b_3 . Thus we have exactly the same conditions as those used in the last paragraph of the proof of Lemma 2.6 to conclude that $\overline{b_3 r_3}$ is an interior diagonal of a solid quadrilateral, hence an outside obstacle is insufficient in this case too.

Therefore, in a 1-obstacle representation of $K_{5,5}^*$, $\text{conv}(B)$ and $\text{conv}(R)$ are disjoint. \square

Armed with the knowledge that every 1-obstacle representation of $K_{5,5}^*$ is an outside obstacle representation and requires R and B to be linearly separable, assume for contradiction that we are given a drawing of $K_{5,5}^*$ that admits a 1-obstacle representation. We will argue that such a drawing necessarily contains a drawing of $K_{2,2}$ requiring more than one obstacle or a drawing of $K_{3,3}$ requiring more than one obstacle. We justify the existence of such a forbidden configuration by using an algorithm that removes vertices from the drawing until casually inspecting the convex hull boundary of the vertices must reveal the existence of such a configuration.

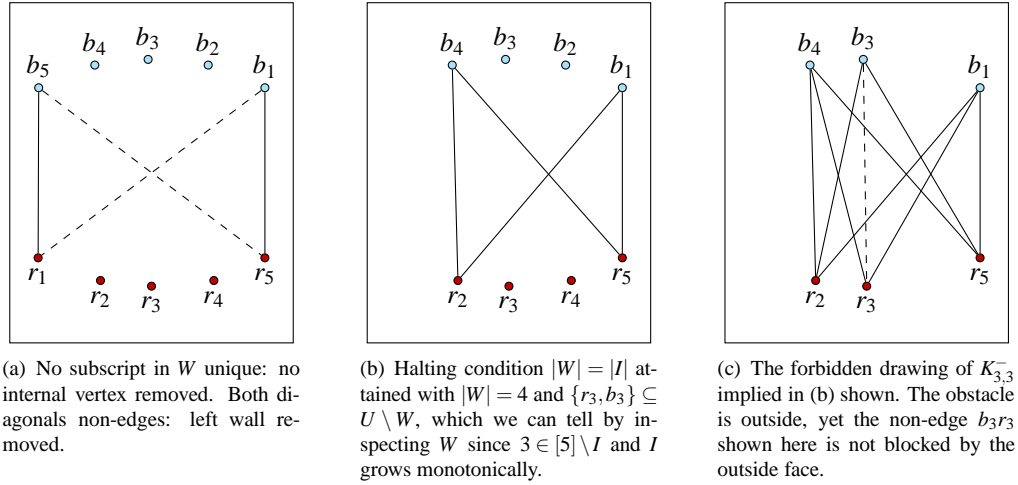
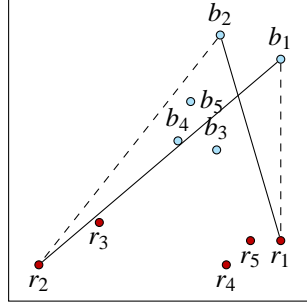


Figure 3: A run of the algorithm in the proof of Lemma 2.1. Notice that the initial state features a placement of $V(K_{5,5}^*)$ in which R is linearly separable from B and below B as required. In all but the last subfigure, only the dichromatic pairs induced on W are shown.

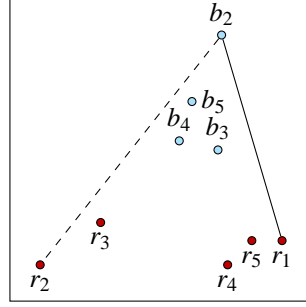
Now we give some terminology needed to describe the algorithm. By Lemma 2.7 the convex hulls of B and R are disjoint, so let the x -axis separate B and R with B above it and R below it. Then for $U \subseteq V(K_{5,5}^*)$ s.t. $|U \cap R| \geq 3$ and $|U \cap B| \geq 3$, in a clockwise walk around the boundary of $\text{conv}(U)$ there is a unique clockwise-ordered pair of consecutive vertices of the form (r_i, b_j) and a unique clockwise-ordered pair of consecutive vertices of the form (b_k, r_ℓ) by the general position assumption. Call $\{r_i, b_j\}$ the left wall of U and denote it by $w_{\text{left}} = w_{\text{left}}(U)$. Likewise, call $\{b_k, r_\ell\}$ the right wall of U and denote it by $w_{\text{right}} = w_{\text{right}}(U)$. Let $W = W(U) = \{r_i, b_j, b_k, r_\ell\}$, the wall vertices of U . The assumptions on U imply $3 \leq |W| \leq 4$. Denote by $I = I(U)$ the set of subscripts occurring in $W(U)$. Then $2 \leq |I| \leq 4$. Observe that $|W| - |I|$ is the number of dichromatic non-edges of $K_{5,5}^*$ induced on W .

Here is the algorithm sketch. Initialize $U := V(K_{5,5}^*)$. The halting condition is $|W| = |I|$, i.e., that every vertex in W has a distinct subscript. Repeat the following until the halting condition arises. For every $i \in [5]$, we say that r_i and b_i are twins. For every vertex with a unique subscript in W , remove its twin from U (unless it has already been removed). Remove at least one vertex in W with a twin also in W , the specifics to be described later.

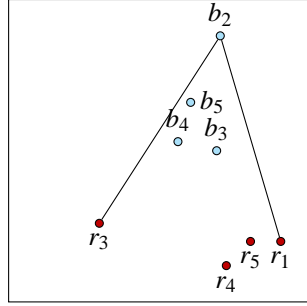
A vertex v is removed from U only if its twin \bar{v} is in W , and removing v will cause \bar{v} to be locked in W for the rest of the algorithm execution, due to the careful way in which we remove a wall vertex. Assuming that this claim holds, I grows monotonically. This means $\{r_j, b_j\} \subseteq U \setminus W$ for every $j \in [5] \setminus I$. Let us call the vertices in $U \setminus W$ the interior vertices of U , and a pair $\{r_j, b_j\} \subseteq U \setminus W$ an interior non-edge of U . Since $|I| \leq |W| \leq 4$ and I grows monotonically, U always has some interior non-edge. Furthermore, at most two vertices from each color class are ever removed,



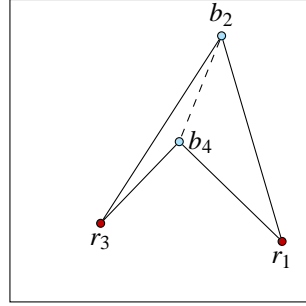
(a) $|W| = 4$ and w_{right} is a non-edge: Removing b_1 will not evict r_1 from the right wall, so b_1 is removed.



(b) $|W| = 3$ and w_{left} is the unique dichromatic non-edge induced on W . The vertex b_2 is kept since it is in both walls, while its twin r_2 is removed.



(c) Halting condition $|W| = |I|$ attained with $|W| = 3$ and $b_4 \in U \setminus W$, which we can tell by inspecting W since $4 \in [5] \setminus I$ and I grows monotonically.



(d) The forbidden drawing of $K_{2,2}$ implied in (c) shown. The obstacle is outside, yet the non-edge b_2b_4 is not blocked by the outside face.

Figure 4: Another run of the algorithm in the proof of Lemma 2.1, illustrating configurations distinct from those shown in Fig. 3.

ensuring the propagation of the precondition $|U \cap R| \geq 3$ and $|U \cap B| \geq 3$ and proper termination. We now show why the halting condition implies a forbidden configuration. The halting condition $|W| = |I|$ arises in two cases:

1. $|W| = 3$. Without loss of generality, $W = \{b_1, r_2, b_3\}$. Then r_4 is an interior vertex of U by the monotonicity of I . We will show that the copy of $K_{2,2}$ induced on $\{b_1, r_2, b_3, r_4\}$ gives a contradiction. r_4 is inside $\text{conv}(\angle b_1 r_2 b_3)$, hence the drawing of the $K_{2,2}$ induced on $\{b_1, r_2, b_3, r_4\}$ is no bowtie, which by Lemma 2.3 yields a contradiction.
2. $|W| = 4$. Without loss of generality, $w_{\text{left}} = \{r_1, b_2\}$ and $w_{\text{right}} = \{b_3, r_4\}$. Then $\{b_5, r_5\}$ is an interior non-edge of U by the monotonicity of I . We will show that the copy of $K_{3,3}^-$ induced on $\{r_1, r_4, r_5, b_2, b_3, b_5\}$ gives a contradiction. Notice that $K_{2,3}$ is induced on $\{r_1, r_4, b_2, b_3, b_5\}$ and on $\{r_1, r_4, r_5, b_2, b_3\}$. Clearly b_2 is the leftmost vertex in the r_1 -sight ordering of $\{b_2, b_3, b_5\}$, r_1 is the rightmost vertex in the b_2 -sight ordering of $\{r_1, r_4, r_5\}$, b_3 is the rightmost vertex in the r_4 -sight ordering of $\{b_2, b_3, b_5\}$, and r_4 is the leftmost vertex in the b_3 -sight ordering of $\{r_1, r_4, r_5\}$. By applying Lemma 2.2 to the aforementioned two vertex sets on which $K_{2,3}$ is induced, we obtain that b_2 and b_3 both see r_5 between r_1 and r_4 , and that r_1 and r_4 both see b_5 between b_2 and b_3 . These conditions are sufficient to ensure that $b_5 r_5$ is an internal diagonal of a solid quadrilateral and hence cannot be blocked by the outside face, contradicting Lemma 2.4.

Now we describe how to remove wall vertices in a way that guarantees the “locking” described above, and hence the monotonicity of I . Note that removing a vertex does not affect a wall that it is not in.

If $|W| = 4$, $w_{\text{left}} = \{r_i, b_j\}$, and $w_{\text{right}} = \{b_k, r_\ell\}$, then we call $\{r_i, b_k\}$ and $\{b_j, r_\ell\}$ the diagonals of U . If both diagonals of U are non-edges, remove from U both vertices in w_{left} . If a single diagonal of U is a non-edge, then without loss of generality, $w_{\text{left}} = \{r_1, b_2\}$ and $w_{\text{right}} = \{b_1, r_3\}$. In this case, proceed to the next iteration by removing b_1 from U . Now we argue why this ensures that r_3 gets locked in the right wall. For every $i \in \{4, 5\}$, $K_{2,2}$ is induced on $\{b_1, r_3, b_2, r_i\}$, hence by Lemma 2.3, $r_i \notin \text{int} \angle b_2 r_3 b_1$. Recalling that r_2 has already been removed, the next counterclockwise vertex after r_3 on the resulting convex hull boundary after removing b_1 will still be blue. Therefore, r_3 remains in the right wall.

If some wall is a non-edge, then without loss of generality, $w_{\text{right}} = \{b_1, r_1\}$. If $|W| = 3$, Without loss of generality, $w_{\text{left}} = \{r_2, b_1\}$. Remove r_1 from U , so that r_2 and b_1 will be locked in w_{left} . If $|W| = 4$, pick the vertex to remove from w_{right} in the following way. If $r_1 \in w_{\text{right}}(U \setminus \{b_1\})$ then remove b_1 , otherwise remove r_1 . To show why this simple action guarantees that the twin of the removed vertex ‘stays’ in the right wall, we need to justify that if $r_1 \notin w_{\text{right}}(U \setminus \{b_1\})$ then $b_1 \in w_{\text{right}}(U \setminus \{r_1\})$.

By hypothesis, $w_{\text{right}}(U \setminus \{b_1\}) = \{b', r'\}$ where $r' \in R \setminus \{r_1\}$. First we must explain why b_1 is the unique blue vertex in U to the right of the line $\overleftrightarrow{r'b'}$. By the definition of right wall, no vertex in $U \setminus \{b_1\}$ is to the right of the line $\overleftrightarrow{r'b'}$. But if b_1 were also to the left of the line $\overleftrightarrow{r'b'}$, then r' together with b' would constitute the right wall of U , contradicting $\{b_1, r_1\} = w_{\text{right}}(U)$. Therefore, b_1 is the unique blue vertex to the right of $\overleftrightarrow{r'b'}$. Initialize a dynamic line L to $\overleftrightarrow{r'b'}$. Rotate L clockwise around $\text{conv}(U \setminus \{b_1, r_1\})$ until it becomes horizontal, allowing it to sweep the entire portion of the half-plane above the x -axis to the right of $\overleftrightarrow{r'b'}$. Clearly, b_1 is the unique blue vertex of U swept by L . Denote by \hat{r} the other vertex of $U \setminus \{r_1\}$ on L at the precise moment when b_1 is swept by L , which is unique by the general position assumption. It is possible that $\hat{r} = r'$. No vertex of $U \setminus \{r_1\}$ is to the right of the line $\overleftrightarrow{\hat{r}b_1}$. Therefore, $\{\hat{r}, b_1\} = w_{\text{right}}(U \setminus \{r_1\})$.

This completes an informal and yet complete specification of the algorithm that shows that every 1-obstacle representation of $K_{5,5}^*$ has a forbidden configuration of vertices resulting in a contradiction. Therefore, the obstacle number of G'_1 , i.e., $K_{5,5}^*$, is greater than one. This implies that the obstacle number of G'_1 is two, per its obstacle representation in Fig. 1. □

3 A 70-vertex $(2, 1)$ -colorable graph without a 1-obstacle representation

Theorem 3.1. *The $(2, 1)$ -colorable graph $G'_2 := CE(6)$, consisting of a clique of 6 blue vertices and an independent set of 64 red vertices each of which has a distinct set of neighbors, has obstacle number greater than one.*

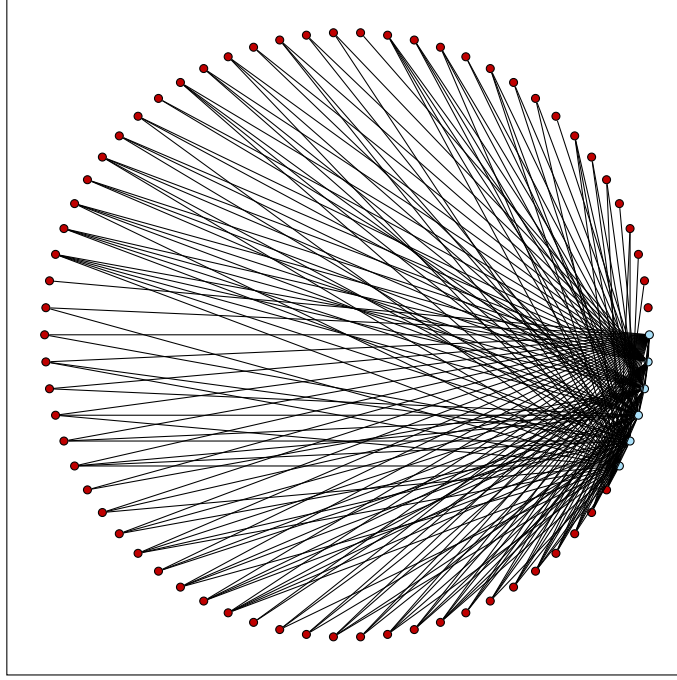


Figure 5: A drawing of G'_2 , i.e., $CE(6)$, whose vertex set consists of a clique (light blue) of six vertices and an independent set (dark red) of 64 vertices with distinct neighborhoods.

Proof. While the graph $CE(6)$ is defined unambiguously by the theorem statement, we give the following definition of the graph family $CE(k)$ in order to assign unique names to the vertices of $CE(6)$, and to be able to refer to its induced subgraphs. Denote by $[k]$ the set of integers $\{1, 2, \dots, k\}$. For $k \in \mathbb{Z}^+$, let $B(k) = \{b_1, b_2, \dots, b_k\}$ be a set of k light blue vertices, and let $R(k) = \{r_A \mid A \subseteq [k]\}$ be a set of 2^k dark red vertices. Let $CE(k)$ be the graph on $B(k) \uplus R(k)$ in which $B(k)$ is a clique, $R(k)$ is an independent set, and there is an edge between $b_i \in B(k)$ and $r_A \in R(k)$ if and only if $i \in A$.

First we present lemmas regarding 1-obstacle representations of $CE(4)$ that will prove instrumental in showing that $CE(6)$ does not have a 1-obstacle representation. We do this by exploiting the hereditary nature of the CE family, that is, whenever $k' < k$, copies of $CE(k')$ can be found as an induced subgraph of $CE(k)$ in a color-preserving fashion.

We first establish some properties in 1-obstacle representations of small CE graphs.

When considering obstacle representations for $CE(k)$, for a fixed index set $A \subseteq [k]$ we denote $[k] \setminus A$ by \bar{A} . For a fixed placement of the blue vertices $B(k)$ in general position, we say a point p in general position with respect to $B(k)$ is A -fragmented if there are distinct $i_1, i_2 \in A$ and distinct $i_3, i_4 \in \bar{A}$ such that $\angle b_{i_1} p b_{i_2}$ separates b_{i_3} from b_{i_4} .

Lemma 3.2. *For every integer $k \geq 4$, every obstacle representation of $CE(k)$ in which some $r_A \in R(k)$ is A -fragmented involves at least two obstacles.*

Proof. For an arbitrary $k \geq 4$, consider an obstacle representation of $CE(k)$ in which for a certain $A \subseteq [k]$, r_A is A -fragmented. Without loss of generality $1, 2 \in A$ and $3, 4 \in \bar{A}$ with $b_3 \in \text{conv}(\angle b_1 r_A b_2)$ and $b_4 \notin \text{conv}(\angle b_1 r_A b_2)$. (See Fig. 6.)

Then the quadrilateral $Q = b_1 r_A b_2 b_3$ is non-self-intersecting and has $r_A b_3$ as an internal diagonal. Hence, an obstacle is needed inside Q , which is interior-disjoint from the complement of $\text{conv}(\angle b_1 r_A b_2)$, in order to block $r_A b_3$. Since $r_A \notin \text{conv}(\angle b_1 r_A b_2)$, so is $r_A b_4$, therefore a different obstacle must block $r_A b_4$. \square

To simplify the notation for red vertices, from now on we will write the subscript i instead of $\{i\}$, and \bar{i} instead of $[k] \setminus \{i\}$ whenever convenient. We will also write B instead of $B(k)$ where the value of k is clear from context.

Lemma 3.3. *For every integer $k \geq 4$, in every 1-obstacle representation of $CE(k)$, B is in convex position.*

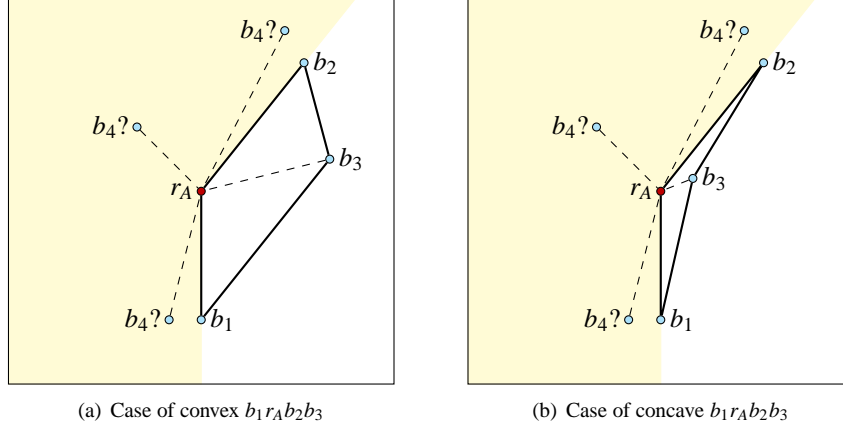


Figure 6: For the proof of Lemma 3.2. The red vertex r_A is A -fragmented with $1, 2 \in A$ and $3, 4 \in \bar{A}$. Without loss of generality, b_3 is in $\text{conv}(\angle b_1r_Ab_2)$ (unshaded) while b_4 is in the complement of $\text{conv}(\angle b_1r_Ab_2)$ (shaded).

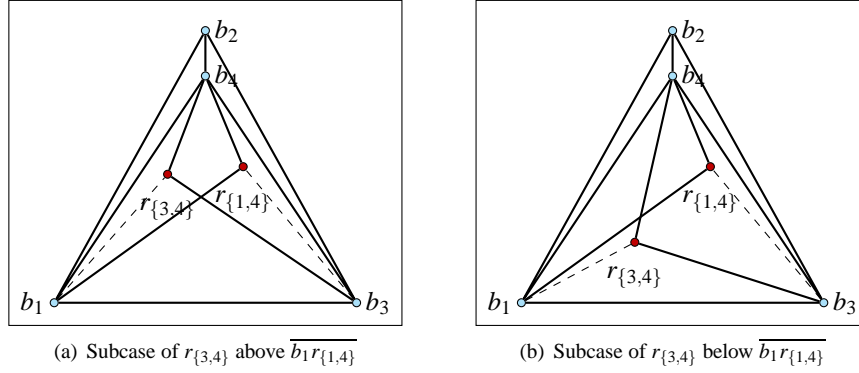


Figure 7: For the proof of Lemma 3.3, Case 1.

Proof. By Carathéodory's Theorem, it is sufficient to prove the result for $k = 4$.

Assume for contradiction that we are given a 1-obstacle representation of $CE(4)$ in which B is not in convex position. Without loss of generality, b_4 is inside the triangle $\Delta b_1b_2b_3$. There are two cases to consider.

Case 1: The obstacle is in $\text{conv}(B)$. Without loss of generality, the obstacle is inside $\Delta b_1b_4b_3$. The vertex $r_{\{1,4\}}$ has non-edges to b_2 and b_3 , so if it were outside of $\Delta b_1b_2b_3$ then at least one of these two non-edges would be outside of $\Delta b_1b_2b_3$, requiring a second obstacle. Nor can $r_{\{1,4\}}$ be inside $\Delta b_1b_2b_4$ or $\Delta b_2b_3b_4$, since that would cause its non-edge with b_2 to be inside that triangle, again requiring a second obstacle. A symmetric argument applies to $r_{\{3,4\}}$. Notice that $r_{\{1,4\}} \in \text{conv}(\angle b_4b_2b_3)$, lest it be $\{1,4\}$ -fragmented. Likewise, $r_{\{3,4\}} \in \text{conv}(\angle b_1b_2b_4)$, lest it be $\{3,4\}$ -fragmented. Then without loss of generality, $r_{\{1,4\}}$ is inside $\Delta r_{\{3,4\}}b_4b_3$ which $b_1r_{\{3,4\}}$ is outside of. (See Fig. 7.) Hence, distinct obstacles are required to block $b_1r_{\{3,4\}}$ and $b_3r_{\{1,4\}}$, a contradiction.

Case 2: The obstacle is outside of $\text{conv}(B)$. Then $r_{\overline{4}} \notin \text{conv}(B)$ and without loss of generality, $r_{\overline{4}} \in \text{conv}(\angle b_1b_4b_3)$. Hence the obstacle is inside $\Delta b_1b_3r_{\overline{4}}$. Since the quadrilateral $Q = b_1b_4b_3r_{\overline{4}}$ is convex, every point outside of Q has a segment joining it to b_4 or $r_{\overline{4}}$ without crossing Q . Therefore, every remaining red vertex without an edge to b_4 , in particular, $r_{\{1,3\}}$, is inside Q . The introduction of $r_{\{1,3\}}$ into the drawing results in interior-disjoint quadrilaterals $Q' = b_1r_{\{1,3\}}b_3r_{\overline{4}}$ and $Q'' = b_1r_{\{1,3\}}b_3b_4$. (See Fig. 8.) Since $r_{\{1,3\}}r_{\overline{4}}$ is inside Q' and $r_{\{1,3\}}b_4$ is inside Q'' , distinct obstacles are required to block these non-edges, a contradiction. \square

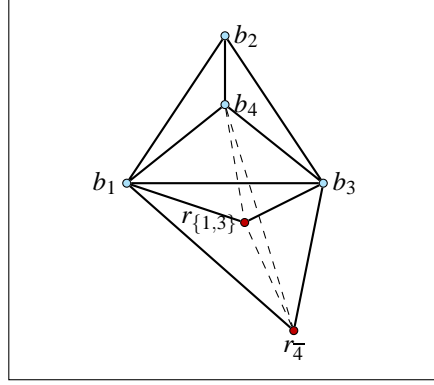


Figure 8: For the proof of Lemma 3.3, Case 2. The assumptions lead without loss of generality to the configuration shown here, with $r_{\{1,3\}}b_4$ and $r_{\{1,3\}}r_4$ requiring distinct obstacles.

Now that we have some restrictions on the relative positions of blue vertices in all 1-obstacle representations of $CE(k)$ for all $k \geq 4$, we pursue the question of where the red vertices can be positioned with respect to the blue vertices.

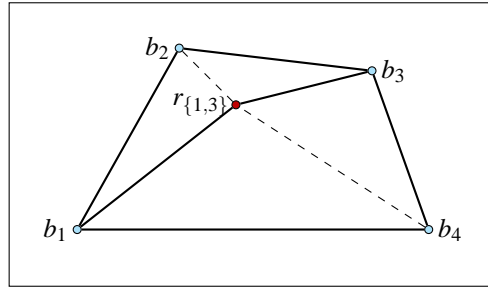


Figure 9: For the proof of Lemma 3.4, Case 2. The vertex $r_{\{1,3\}}$ is $\{1, 3\}$ -fragmented.

Lemma 3.4. *For every integer $k \geq 4$, in every 1-obstacle representation of $CE(k)$ the obstacle is outside of $\text{conv}(B)$, and hence $R \setminus \{r_B\}$ is outside of $\text{conv}(B)$.*

Proof. It is clearly enough to establish the lemma in the special case $k = 4$.

Assume for contradiction that we are given a 1-obstacle representation of $CE(4)$ such that the obstacle is in $\text{conv}(B)$. By Lemma 3.3, B is in convex position. Without loss of generality, $b_1b_2b_3b_4$ is a clockwise enumeration of B .

Case 1: $r_{\{1,3\}} \notin \text{conv}(B)$. Imagine the polygon $b_1b_2b_3b_4$ bounding $\text{conv}(B)$ as opaque: Since it is convex, $r_{\{1,3\}}$ sees some side of $\text{conv}(B)$ in its entirety, hence $r_{\{1,3\}}$ sees b_i for a certain even i . This means $r_{\{1,3\}}b_i$ is outside $\text{conv}(B)$, hence it will require a separate obstacle, a contradiction.

Case 2: $r_{\{1,3\}} \in \text{conv}(B)$. (See Fig. 9.) By the convexity of B , $r_{\{1,3\}}$ is $\{1, 3\}$ -fragmented, a contradiction. \square

We introduce some further terminology to use in the context of $CE(k)$ (for any integer $k > 0$) for a fixed arrangement of B . The following definitions are meant only for points outside of $\text{conv}(B)$ and in general position with respect to B .

For a given $A \subseteq [k]$, let B_A denote $\{b_i \mid i \in A\}$. We say that a point p is A -straight if some line through p separates B_A and $B_{\bar{A}}$ (vacuously true if $A \in \{\emptyset, [k]\}$). We say that a point p is A -convex if it sees $B_A \neq \emptyset$ between two non-empty parts of $B_{\bar{A}}$ that comprise $B_{\bar{A}}$. If a point is \bar{A} -convex, we say it is A -reflex. Observe that if r_A is A -reflex, then an

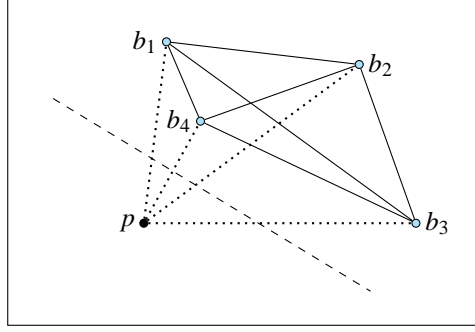


Figure 10: Illustration for the concepts *A-straight*, *A-convex*, and *A-reflex*. For $CE(4)$ or $CC(4)$, consider the given placement of B . The point p is $\{2, 3\}$ -straight, $\{4\}$ -convex, and $\{1, 4, 3\}$ -reflex.

obstacle is required in a bounded face, but not necessarily in the unbounded face. Note that for every $A \subseteq [k]$, every point $p \notin \text{conv}(B)$ in general position with respect to B is either *A-straight*, *A-convex*, *A-reflex*, or *A-fragmented*. (See Fig. 10.)

Now we can finish proving with relative ease that $CE(6)$ does not admit a 1-obstacle representation. Assume for contradiction that we are given a 1-obstacle representation of $CE(6)$. By Lemma 3.3, B is in convex position. Without loss of generality, $b_1 b_2 b_3 b_4 b_5 b_6$ is a clockwise enumeration of B . By Lemma 3.4, $R \setminus \{r_B\}$ is outside of $\text{conv}(B)$. In particular, $r_{\{1,3,5\}}$ is outside of $\text{conv}(B)$. We will show that every point outside of $\text{conv}(B)$ and in general position with respect to B is $\{1, 3, 5\}$ -fragmented by showing that it is neither $\{1, 3, 5\}$ -straight nor $\{1, 3, 5\}$ -convex nor $\{1, 3, 5\}$ -reflex.

Clearly, no point is $\{1, 3, 5\}$ -straight, since $\{b_1, b_3, b_5\}$ is not linearly separable from $\{b_2, b_4, b_6\}$.

Assume for contradiction that some point p is $\{1, 3, 5\}$ -convex. Hence p sees odd-subscripted blue vertices together between two sets of even-subscripted blue vertices. Then p is $\{i, j\}$ -straight for some $\{i, j\} \subseteq \{2, 4, 6\}$. But $b_i b_j$ is a diagonal of the bounding hexagon of B , which contradicts that it is linearly separable from $B \setminus \{b_i, b_j\}$. By a symmetric argument, no point is $\{2, 4, 6\}$ -convex (i.e., $\{1, 3, 5\}$ -reflex) either. Therefore, $r_{\{1,3,5\}}$ is $\{1, 3, 5\}$ -fragmented, requiring

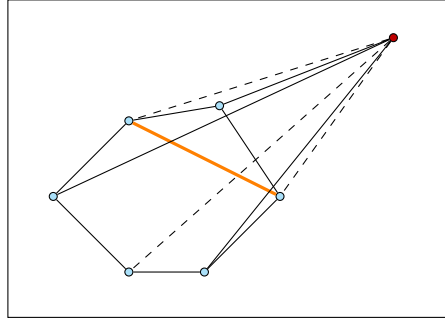


Figure 11: For the proof of Theorem 3.1. A red vertex r_A adjacent exactly to blue vertices non-adjacent in the bounding polygon of B is *A-fragmented* no matter what, as in this example.

two obstacles, a contradiction.

Therefore, G'_2 , i.e., $CE(6)$, has obstacle number greater than one. □

4 A 10-vertex $(2,2)$ -colorable graph without a 1-obstacle representation

We showed in [8] that a $(2,2)$ -colorable 20-vertex graph G_3 has obstacle number greater than 1. One can obtain G_3 from $CE(4)$ by adding all possible edges among the vertices in the independent set of 16 red vertices. Here, we show that a 10-vertex induced subgraph of it, G'_3 , also has obstacle number greater than 1.

Let G'_3 be the graph consisting of a clique of light blue vertices $B = \{b_i \mid i \in [4]\}$, a clique of dark red vertices $R = \{r_A \mid A \in \binom{[4]}{2}\}$, and additional edges between every b_i and every r_A with $i \in A$. (See Fig. 12.)

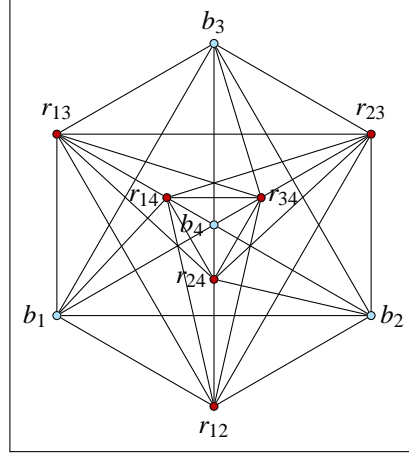


Figure 12: A drawing of G'_3 with 3-fold rotational symmetry.

Theorem 4.1. G'_3 , a $(2,2)$ -colorable graph on ten vertices, has obstacle number greater than one.

Proof. We say that a polygon is *solid* if all its edges are edges in G'_3 . For three distinct points p , q , and r , we denote by $\angle pqr$ the union of the rays \overrightarrow{qp} and \overrightarrow{qr} . For a point set P , we denote by $\text{conv}(P)$ the convex hull of P (the smallest convex set containing P).

Assume for contradiction that we are given a 1-obstacle representation of G'_3 . Following the terminology in the preceding section, we shall say that a red vertex r_A is *fragmented* if it is *A-fragmented*. That is, a vertex r_A is not fragmented if and only if there are points p and q such that $\angle pr_Aq$ strictly separates $\{b_i \mid i \in A\}$ from the remaining blue vertices. If some red vertex r_A is fragmented, then *two* obstacles will be required due to $\{r_A\} \cup B$, a contradiction.

Case 1: B is not in convex position. Without loss of generality, b_4 is inside the triangle $\Delta b_1 b_2 b_3$.

Subcase 1a: The obstacle is in $\text{conv}(B)$. The proof of Lemma 3.3 Case 1 is based only on the vertices $b_1, b_2, b_3, b_4, r_{\{1,4\}}$ and $r_{\{2,4\}}$, under the same conditions, hence that argument applies verbatim to yield a contradiction here.

Subcase 1b: The obstacle is outside of $\text{conv}(B)$. Let $C_{\{1,2\}} = \text{conv}(\angle b_2 b_4 b_1)$, $C_{\{1,3\}} = \text{conv}(\angle b_1 b_4 b_3)$, and $C_{\{2,3\}} = \text{conv}(\angle b_3 b_4 b_2)$. Every red vertex is in precisely one of these regions and outside of $\text{conv}(B)$. Let $f : \binom{[3]}{2} \rightarrow \binom{[3]}{2}$ be the map such that $r_A \in C_{f(A)}$ whenever $A \in \binom{[3]}{2}$. We will show that every possible assumption about f leads to a contradiction.

Assume for contradiction that f has a fixed point. Without loss of generality, $r_{\{1,2\}} \in C_{\{1,2\}}$. This means that $Q = b_2 b_4 b_1 r_{\{1,2\}}$ is a solid convex quadrilateral, hence to block $b_4 r_{\{1,2\}}$, the obstacle is inside Q . Then, $r_{\{3,4\}}$ must be inside Q in order for the obstacle to block both $b_1 r_{\{3,4\}}$ and $b_2 r_{\{3,4\}}$. But then, $\angle b_4 r_{\{3,4\}} r_{\{1,2\}}$ partitions Q into disjoint quadrilateral regions with solid boundaries that respectively contain $b_1 r_{\{3,4\}}$ and $b_2 r_{\{3,4\}}$. Hence, two obstacles are required, a contradiction. Therefore, f has no fixed point.

Assume for contradiction that f is not a permutation. Without loss of generality, $r_{\{1,3\}}$ and $r_{\{2,3\}}$ are both in $C_{\{1,2\}}$. In order for both of these red vertices to not be fragmented, $\overleftrightarrow{b_3 b_4}$ must separate $b_1 r_{\{1,3\}}$ and $b_2 r_{\{2,3\}}$. Hence, $Q = b_2 b_1 r_{\{1,3\}} r_{\{2,3\}}$ is a solid, non-self-intersecting quadrilateral. If Q is concave, we get an immediate contradiction due to Q separating its diagonals, both of which are non-edges in G'_3 . If Q is convex, the obstacle is inside Q in order

to block its diagonals. But since $r_{\{1,2\}}$ is outside of $C_{\{1,2\}}$, it does not meet $\text{conv}(Q)$, requiring another obstacle, a contradiction. Therefore, f is a permutation.

Since f is a permutation of three elements with no fixed point, it is cyclic. Without loss of generality, $r_{\{1,2\}} \in C_{\{2,3\}}$ and $r_{\{1,3\}} \in C_{\{1,2\}}$. In order to not be fragmented, $r_{\{1,2\}}$ is on the same side of $\overleftrightarrow{b_1 b_4}$ as b_2 , and $r_{\{1,3\}}$ is on the same side of $\overleftrightarrow{b_3 b_4}$ as b_1 . These conditions ensure that $b_1 b_4$ does not meet $r_{\{1,2\}} r_{\{1,3\}}$. If $b_2 b_4$ and $r_{\{1,2\}} r_{\{1,3\}}$ meet at some point p , then the convex solid quadrilateral $b_1 r_{\{1,3\}} p b_4$ will have $b_2 r_{\{1,3\}}$ inside and $b_4 r_{\{1,2\}}$ outside, requiring two obstacles, a contradiction. If not, then $b_1 r_{\{1,3\}} r_{\{1,2\}} b_2 b_4$ is a non-self-intersecting solid pentagon with $b_2 r_{\{1,3\}}$ inside and $b_4 r_{\{1,2\}}$ outside, requiring two obstacles, a contradiction.

Having exhausted all possibilities, we have shown that the assumptions of Subcase 1b lead to a contradiction.

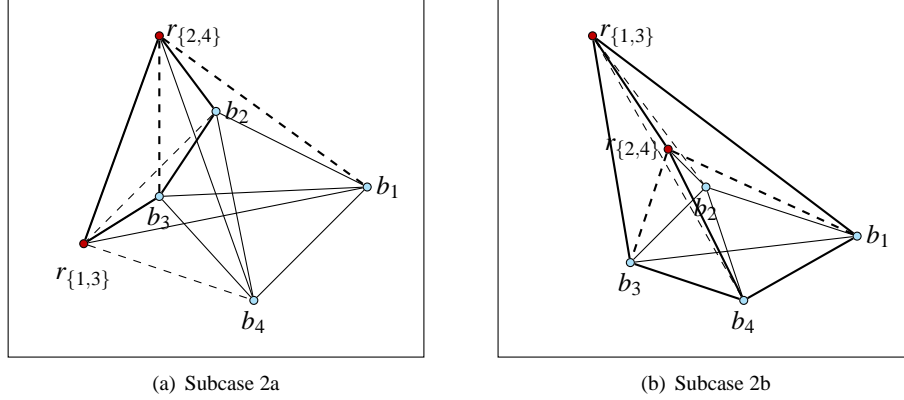


Figure 13: For the proof of Theorem 4.1 Case 2. The thick dashed non-edges require distinct obstacles.

Case 2: B is in convex position. Without loss of generality, the bounding polygon of B is $b_1 b_2 b_3 b_4$. To not be fragmented,

- (i) $r_{\{1,3\}}$ and $r_{\{2,4\}}$ must lie outside of $\text{conv}(B)$;
- (ii) for $r_{\{1,3\}}$, either $b_1, b_3 \in \text{conv}(\angle b_2 r_{\{1,3\}} b_4)$ or $b_2, b_4 \in \text{conv}(\angle b_1 r_{\{1,3\}} b_3)$; and
- (iii) for $r_{\{2,4\}}$, either $b_1, b_3 \in \text{conv}(\angle b_2 r_{\{2,4\}} b_4)$ or $b_2, b_4 \in \text{conv}(\angle b_1 r_{\{2,4\}} b_3)$.

Subcase 2a: $b_1, b_3 \in \text{conv}(\angle b_2 r_{\{1,3\}} b_4)$ and $b_2, b_4 \in \text{conv}(\angle b_1 r_{\{2,4\}} b_3)$. Without loss of generality, the quadrilateral $b_4 b_1 b_2 r_{\{1,3\}}$ is convex and has b_3 inside, and without loss of generality, the quadrilateral $b_3 b_4 b_1 r_{\{2,4\}}$ is convex and has b_2 inside. Hence, $b_2 b_3 r_{\{1,3\}} r_{\{2,4\}}$ is a solid convex quadrilateral with $b_1 r_{\{2,4\}}$ outside and $b_3 r_{\{2,4\}}$ inside. Therefore, two obstacles are required, a contradiction.

Subcase 2b: $b_2, b_4 \in \text{conv}(\angle b_1 r_{\{1,3\}} b_3)$ or $b_1, b_3 \in \text{conv}(\angle b_2 r_{\{2,4\}} b_4)$. Due to symmetry, we proceed assuming the former. Without loss of generality, $Q = b_3 b_4 b_1 r_{\{1,3\}}$ is a convex quadrilateral. The obstacle is inside Q due to $r_{\{1,3\}} b_4$. In order for $b_1 r_{\{2,4\}}$ and $b_3 r_{\{2,4\}}$ to be blocked, $r_{\{2,4\}}$ is inside Q . Hence, $\angle r_{\{1,3\}} r_{\{2,4\}} b_4$ partitions $\text{conv}(Q)$ into two regions with solid boundaries that respectively contain $b_1 r_{\{2,4\}}$ and $r_{\{2,4\}} b_3$. Therefore, two obstacles are required, a contradiction.

Therefore, G'_3 has obstacle number greater than one. □

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